

Quantum versus Classical Domains for Teleportation with Continuous Variables

Samuel L. Braunstein[†], Christopher A. Fuchs[‡], H. J. Kimble*, and P. van Loock[†]

[†]*Informatics, Bangor University, Bangor LL57 1UT, UK*

[‡]*Bell Labs, Lucent Technologies, 600-700 Mountain Ave., Room 2C-420, Murray Hill, NJ 07974, USA*

**Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125, USA*

(February 1, 2008)

Fidelity $F_{\text{classical}} = \frac{1}{2}$ has been established as setting the boundary between classical and quantum domains in the teleportation of coherent states of the electromagnetic field (S. L. Braunstein, C. A. Fuchs, and H. J. Kimble, *J. Mod. Opt.* **47**, 267 (2000)). Two recent papers by P. Grangier and F. Grosshans ([quant-ph/0009079](#) and [quant-ph/0010107](#)) introduce alternate criteria for setting this boundary and as a result claim that the appropriate boundary should be $F = \frac{2}{3}$. Although larger fidelities would lead to enhanced teleportation capabilities, we show that the new conditions of Grangier and Grosshans are largely unrelated to the questions of entanglement and Bell-inequality violations that they take to be their primary concern. With regard to the quantum-classical boundary, we demonstrate that fidelity $F_{\text{classical}} = \frac{1}{2}$ remains the appropriate point of demarcation. The claims of Grangier and Grosshans to the contrary are simply wrong, as we show by an analysis of the conditions for nonseparability (that complements our earlier treatment) and by explicit examples of Bell-inequality violations.

I. INTRODUCTION

As proposed by Bennett *et al.* [1], the protocol for achieving quantum teleportation is the following. Alice is to transfer an unknown quantum state $|\psi\rangle$ to Bob, using as the sole resources some previously shared *quantum entanglement* and a *classical channel* capable of communicating measurement results. Physical transport of $|\psi\rangle$ from Alice to Bob is excluded at the outset. Ideal teleportation occurs when the state $|\psi\rangle$ enters Alice's sending station and the *same* state $|\psi\rangle$ emerges from Bob's receiving station.

Of course, in actual experiments [2–5], the ideal case is *unattainable* as a matter of principle. The question of operational criteria for gauging success in an experimental setting, therefore, cannot be avoided. We have proposed previously that a minimal set of conditions for claiming success in the laboratory are the following [6].

1. An unknown quantum state (supplied by a third party Victor) is input physically into Alice's station from an outside source.
2. The “recreation” of this quantum state emerges from Bob's receiving terminal available for Victor's independent examination.
3. There should be a quantitative measure for the quality of the teleportation and based upon this measure, it should be clear that shared entanglement enables the output state to be “closer” to the input state than could have been achieved if Alice and Bob had utilized a classical communication channel alone.

In Ref. [6], it was shown that the fidelity F between input and output states is an appropriate measure of the

degree of similarity in Criterion 3. For an input state $|\psi_{\text{in}}\rangle$ and output state described by the density operator $\hat{\rho}_{\text{out}}$, the fidelity is given by [7]

$$F = \langle \psi_{\text{in}} | \hat{\rho}_{\text{out}} | \psi_{\text{in}} \rangle. \quad (1)$$

To date only the experiment of Furusawa *et al.* [4] has achieved unconditional experimental teleportation as defined by the three criteria above [6,8,9]. This experiment was carried out in the setting of continuous quantum variables with input states $|\psi_{\text{in}}\rangle$ consisting of coherent states of the electromagnetic field, with an observed fidelity $F_{\text{exp}} = 0.58 \pm 0.02$ having been attained. This benchmark is significant because it can be demonstrated [4,6] that quantum entanglement is the critical ingredient in achieving an average fidelity greater than $F_{\text{classical}} = \frac{1}{2}$ *when* the input is an absolutely random coherent state [10].

Against this backdrop, Grangier and Grosshans [11,12] have recently suggested that the appropriate boundary between the classical and quantum domains in the teleportation of coherent states should be a fidelity $F = \frac{2}{3}$. Their principal concern is the distinction between “entanglement” and “non-separability,” where by the latter term, they mean “the physical properties associated with non-locality and the violation of Bell's inequalities (BI).”^{*} They claim that “due to imperfect transmissions,

^{*}Since the terms “entanglement” and “nonseparability” are used interchangeably in the quantum information community, we will treat them as synonyms to eliminate further confusion. We will refer to violations of Bell's inequalities explicitly whenever a distinction must be made between entanglement and local realism *per se*. The only exceptions will be when we quote directly from Grangier and Grosshans [11,12].

... it becomes possible to violate the classical boundary (*i.e.*, $F = \frac{1}{2}$) of teleportation without any violation of BI.” [11] However, rather than addressing the issue in a direct manner, they then propose the violation of a certain “Heisenberg-type inequality (HI)” as “a more effective – and in some sense ‘necessary’ – way to characterize shared entanglement.” It is this that leads to their condition $F > \frac{2}{3}$ as being necessary for the declaration of successful teleportation. In support of this threshold, they further relate their criterion based on the HI to ones previously introduced in the quantum nonde-molition measurement (QND) literature. Finally, in Ref. [12], Grangier and Grosshans find that $F > \frac{2}{3}$ is also required by a criterion they introduce having to do with a certain notion of reliable “information exchange” [12].

The purpose of the present paper is to demonstrate that the conclusions of Grangier and Grosshans concerning the proposed quantum-classical boundary $F = \frac{2}{3}$ are unwarranted and, by explicit counter example, incorrect. Our approach will be to investigate questions of nonseparability and violations of Bell inequalities for the particular entangled state employed in the teleportation protocol of Ref. [13]. Of significant interest will be the case with losses, so that the relevant quantum states will be mixed quantum states. Our analysis supports the following conclusions.

1. Although the argument of Grangier and Grosshans is claimed to be based upon “EPR non-separability of the entanglement resource” [12] [by which they mean a potential violation of a BI], they offer no quantitative connection (by constructive proof or otherwise) between the criteria they introduce (including the threshold $F = \frac{2}{3}$) [11,12] and the actual violation of any Bell inequality. Nothing in their analysis provides a warranty that $F > \frac{2}{3}$ would preclude a description in terms of a local hidden-variables theory. They offer only the suggestion that “ $F > \frac{2}{3}$ would be much safer” [11].
2. By application of the work of Duan *et al.* [14], Simon [15], and Tan [16], we investigate the question of entanglement. We show that the states employed in the experiment of Ref. [4] are nonseparable, as was operationally confirmed in the experiment. Moreover, we study the issue of nonseparability for mixed states over a broad range in the degree of squeezing for the initial EPR state, in the overall system loss, and in the presence of thermal noise. This analysis reveals that EPR mixed states that are nonseparable do indeed lead to a fidelity of $F > F_{\text{classical}} = \frac{1}{2}$ for the teleportation of coherent states. Hence, in keeping with Criterion 3 above, the threshold fidelity for employing entanglement as a quantum resource is precisely the same as was deduced in the previous analysis of Ref. [6]. Within

the setting of Quantum Optics, this threshold coincides with the standard benchmark for manifestly quantum or nonclassical behavior, namely that the Glauber-Sudarshan phase-space function becomes nonpositive-definite, here for any bipartite nonseparable state [17]. By contrast, the value $F = \frac{2}{3}$ championed by Grangier and Grosshans is essentially unrelated to the threshold for entanglement (nonseparability) in the teleportation protocol, as well as to the boundary for the nonclassical character of the EPR state.

3. By application of the work of Banaszek and Wodkiewicz [18,19], we explore the possibility of violations of Bell inequalities for the EPR (mixed) states employed in the teleportation of continuous quantum-variables states. We find direct violations of a CHSH inequality [20] over a large domain. Significant relative to the claims of Grangier and Grosshans is a regime both of entanglement (nonseparability) and of violation of a CHSH inequality for which the teleportation fidelity $F < \frac{2}{3}$ and for which the criterion of the Heisenberg inequalities of Ref. [11] fails. Hence, teleportation with $\frac{1}{2} < F < \frac{2}{3}$ is possible with EPR (mixed) states which do not admit a local hidden variables description. In contradistinction to the claim of Grangier and Grosshans, $F > \frac{2}{3}$ does not provide a relevant criterion for delineating the quantum and classical domains with respect to violations of Bell’s inequalities.
4. By adopting a protocol analogous to that employed in *all* previous experimental demonstrations of violations of Bell’s inequalities [21–23], scaled correlation functions can be introduced for continuous quantum variables. In terms of these scaled correlations, the EPR mixed state used for teleportation violates a generalized version of the CHSH inequality, though non-ideal detector efficiencies require a “fair sampling” assumption for this. These violations set in for $F > \frac{1}{2}$ and have recently been observed in a setting of low detection efficiency [24]. This experimental verification of a violation of a CHSH inequality (with a fair sampling assumption) again refutes the purported significance of the threshold $F = \frac{2}{3}$ promoted by Grangier and Grosshans.

Overall, we find no support for the claims of Grangier and Grosshans giving special significance to the threshold fidelity $F = \frac{2}{3}$ in connection to issues of separability and Bell-inequality violations. Instead, as we will show, it is actually the value $F_{\text{classical}} = \frac{1}{2}$ that heralds entrance into the quantum domain with respect to the very same issues. Their claims based upon a Heisenberg-type inequality and a criterion for “information exchange” are

essentially unrelated to the issue of a quantum-classical boundary.

All this is not to say that teleportation of coherent states with increasing degrees of fidelity beyond $F_{\text{classical}} = \frac{1}{2}$ to $F > \frac{2}{3}$ is not without significance. In fact, as tasks of ever increasing complexity are to be accomplished, there will be corresponding requirements to improve the fidelity of teleportation yet further. Moreover, there are clearly diverse quantum states other than coherent states that one might desire to teleport, including squeezed states, quantum superpositions, entangled states [16], and so on. The connection between the “intricacy” of such states and the requisite resources for achieving high fidelity teleportation has been discussed in Ref. [13], including the example of the superposition of two coherent states,

$$|\alpha\rangle + |-\alpha\rangle, \quad (2)$$

which for $|\alpha| \gg 1$ requires an EPR state with an extreme degree of quantum correlation.

Similarly, Heisenberg-type inequalities are in fact quite important for the inference of the properties of a *system* given the outcomes of measurements made on a *meter* following a *system-meter* interaction. Such quantities are gainfully employed in Quantum Optics in many settings, including realizations of the original EPR *gedanken* experiment [25–27] and of back-action evading measurement and quantum non-demolition detection [28].

Our only point is that the claim of Grangier and Grosshans that $F = \frac{2}{3}$ is required for the “successful quantum teleportation of a coherent state” [12] is incorrect. They simply offer no quantitative analysis directly relevant to either entanglement or Bell-inequality violation issues. In contrast, the prior treatment of Ref. [6] demonstrates that in the absence of shared entanglement between Alice and Bob, there is an upper limit for the fidelity for the teleportation of randomly chosen coherent states given by $F_{\text{classical}} = \frac{1}{2}$. Nothing in the work of Grangier and Grosshans calls this analysis into question.

This, however, leads to something we would like to stress apart from the details of any particular teleportation criterion. There appears to be a growing confusion in the community that equates quantum teleportation experiments with fundamental tests of quantum mechanics. The purpose of such tests is generally to compare quantum mechanics to other potential theories, such as locally realistic hidden-variable theories [11,29,30]. In our view, experiments in teleportation have nothing to do with this. They instead represent investigations *within* quantum mechanics, demonstrating only that a particular task can be accomplished with the resource of quantum entanglement and cannot be accomplished without it. This means that violations of Bell’s inequalities are largely irrelevant as far as the original proposal of Bennett *et al.* [1] is concerned, as well as for experimental

implementations of that protocol. In a theory which allows states to be cloned, there would be no need to discuss teleportation at all – unknown states could be cloned and transmitted with fidelity arbitrarily close to one.

These comments notwithstanding, Grangier and Grosshans did nevertheless attempt to link the idea of Bell-inequality violations with the fidelity of teleportation. It is to the details of that linkage that we now turn. The remainder of the paper is organized as follows. In Section II, we extend the prior work of Ref. [6] to a direct treatment of the consequences of shared entanglement between Alice and Bob, beginning with an explicit model for the mixed EPR states used for teleportation of continuous quantum variables. In Section III we review the criteria Grangier and Grosshans introduced in preparation for showing their inappropriateness as tools for the questions at hand. In Section IV, we demonstrate explicitly the relationship between entanglement and fidelity, and find the same threshold $F_{\text{classical}} = \frac{1}{2}$ as in our prior analysis [6]. The value $F = \frac{2}{3}$ is shown to have no particular distinction in this context. In Sections V and VI, we further explore the role of entanglement with regard to violations of a CHSH inequality and provide a quantitative boundary for such violations. Again, $F_{\text{classical}} = \frac{1}{2}$ appears as the point of entry into the quantum domain, with the point $F = \frac{2}{3}$ having no particular distinction. Our conclusions are collected in Section VII. Of particular significance, we point out that the teleportation experiment of Ref. [4] did indeed cross from the classical to the quantum domain, just as advertised previously.

II. THE EPR STATE

The teleportation protocol we consider is that of Braunstein and Kimble [13], for which the relevant entangled state is the so-called two-mode squeezed state. This state is given explicitly in terms of a Fock-state expansion for two-modes (1, 2) by [31,32]

$$|EPR\rangle_{1,2} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n |n\rangle_1 |n\rangle_2, \quad (3)$$

where r measures the amount of squeezing required to produce the entangled state. Note that for simplicity we consider the case of two single modes for the electromagnetic field; the extension to the multimode case for fields of finite bandwidth can be found in Ref. [33].

The pure state of Eq. (3) can be equivalently described by the corresponding Wigner distribution W_{EPR} over the two modes (1, 2),

$$\begin{aligned} & W_{\text{EPR}}(x_1, p_1; x_2, p_2) \\ &= \frac{4}{\pi^2} \frac{1}{\sigma_+^2 \sigma_-^2} \exp \left(-[(x_1 + x_2)^2 + (p_1 - p_2)^2]/\sigma_+^2 \right. \\ & \quad \left. -[(x_1 - x_2)^2 + (p_1 + p_2)^2]/\sigma_-^2 \right), \end{aligned} \quad (4)$$

where σ_{\pm} are expressed in terms of the squeezing parameter by

$$\begin{aligned}\sigma_+^2 &= e^{+2r}, \\ \sigma_-^2 &= e^{-2r},\end{aligned}\quad (5)$$

with $\sigma_+^2 \sigma_-^2 = 1$. Here, the canonical variables (x_j, p_j) are related to the complex field amplitude α_j for mode $j = (1, 2)$ by

$$\alpha_j = x_j + ip_j. \quad (6)$$

In the limit of $r \rightarrow \infty$, Eq. (4) becomes

$$C \delta(x_1 - x_2) \delta(p_1 + p_2) \quad (7)$$

which makes a connection to the original EPR state of Einstein, Podolsky, and Rosen [25].

Of course, W_{EPR} as given above is for the ideal, lossless case. Of particular interest with respect to experiments is the inclusion of losses, as arise from, for example, finite propagation and detection efficiencies. Rather than deal with any detailed setup (e.g., as treated in explicit detail in Ref. [26]) here we adopt a generic model of the following form. Consider two identical beam splitters each with a transmission coefficient η , one for each of the two EPR modes. We take $0 \leq \eta \leq 1$, with $\eta = 1$ for the ideal, lossless case. The input modes to the beam splitter 1 are taken to be $(1', a')$, while for beam splitter 2, the modes are labeled by $(2', b')$. Here, the modes $(1', 2')$ are assumed to be in the state specified by the ideal W_{EPR} as given in Eq. (4) above, while the modes (a', b') are taken to be independent thermal (mixed) states each with Wigner distribution

$$W(x, p) = \frac{1}{\pi(\bar{n} + \frac{1}{2})} \exp\{-(x^2 + p^2)/(\bar{n} + 1/2)\}, \quad (8)$$

where \bar{n} is the mean thermal photon number for each of the modes (a', b') .

The overall Wigner distribution for the initial set of input modes $(1', 2'), (a', b')$ is then just the product

$$W_{\text{EPR}}(x_{1'}, p_{1'}; x_{2'}, p_{2'}) W(x_{a'}, p_{a'}) W(x_{b'}, p_{b'}) . \quad (9)$$

The standard beam-splitter transformations lead in a straightforward fashion to the Wigner distribution for the output set of modes $(1, 2), (a, b)$, where, for example,

$$\begin{aligned}x_1 &= \sqrt{\eta} x_{1'} - \sqrt{1 - \eta} x_{a'}, \\ x_a &= \sqrt{\eta} x_{a'} + \sqrt{1 - \eta} x_{1'}.\end{aligned}\quad (10)$$

We require $W_{\text{EPR}}^{\text{out}}$ for the $(1, 2)$ modes alone, which is obtained by integrating over the (a, b) modes. A straightforward calculation results in the following distribution for the mixed output state:

$$\begin{aligned}W_{\text{EPR}}^{\text{out}}(x_1, p_1; x_2, p_2) \\ = \frac{4}{\pi^2} \frac{1}{\bar{\sigma}_+^2 \bar{\sigma}_-^2} \exp\left(-[(x_1 + x_2)^2 + (p_1 - p_2)^2]/\bar{\sigma}_+^2 \right. \\ \left. - [(x_1 - x_2)^2 + (p_1 + p_2)^2]/\bar{\sigma}_-^2\right),\end{aligned}\quad (11)$$

where $\bar{\sigma}_{\pm}$ are given by

$$\begin{aligned}\bar{\sigma}_+^2 &= \eta e^{+2r} + (1 - \eta)(1 + 2\bar{n}), \\ \bar{\sigma}_-^2 &= \eta e^{-2r} + (1 - \eta)(1 + 2\bar{n}).\end{aligned}\quad (12)$$

Note that $W_{\text{EPR}}^{\text{out}}$ as above follows directly from W_{EPR} in Eq. (4) via the simple replacements $\sigma_{\pm} \rightarrow \bar{\sigma}_{\pm}$. Relevant to the discussion of Bell inequalities in Sections V and VI is the fact that $\bar{\sigma}_+^2 \bar{\sigma}_-^2 > 1$ for any $r > 0$ and $\eta < 1$.

III. THE CRITERIA OF GRANGIER AND GROSSHANS

The two recent papers of Grangier and Grosshans argue that “fidelity value larger than $\frac{2}{3}$ is actually required for successful teleportation” [11,12]. In this section, we recapitulate the critical elements of their analysis and state their criteria in the present notation. In subsequent sections we proceed further with our own analysis of entanglement and possible violations of Bell’s inequalities for the EPR state of Eq. (11).

Beginning with Ref. [11], Eq. (21), Grangier and Grosshans state the following:

“As a criteria for non-separability [by which they mean violations of Bell’s inequalities], we will use the EPR argument: two different measurements prepare two different states, in such a way that the product of conditional variances (with different conditions) violates the Heisenberg principle.”

This statement takes a quantitative form in terms of the following conditional variances expressed in the notation of the preceding section for EPR beams $(1, 2)$:

$$\begin{aligned}V_{x_i|x_j} &= \langle \Delta x_i^2 \rangle - \frac{\langle x_i x_j \rangle^2}{\langle \Delta x_j^2 \rangle}, \\ V_{p_i|p_j} &= \langle \Delta p_i^2 \rangle - \frac{\langle p_i p_j \rangle^2}{\langle \Delta p_j^2 \rangle}.\end{aligned}\quad (13)$$

with $(i, j) = (1, 2)$ and $i \neq j$. Note that, for example, $V_{x_2|x_1}$ gives the error in the knowledge of the canonical variable x_2 based upon an estimate of x_2 from a measurement of x_1 , and likewise for the other conditional variances. These variances were introduced in Refs. [26,27] in connection with an optical realization of the original *gedanken* experiment of Einstein, Podolsky, and Rosen [25]. An apparent violation of the uncertainty principle arises if the product of inference errors is below the

uncertainty product for one beam alone. For example, $V_{x_2|x_1}V_{p_2|p_1} < \frac{1}{16}$ represents such an apparent violation since $\Delta x_{1,2}^2 \Delta p_{1,2}^2 \geq \frac{1}{16}$ is demanded by the canonical commutation relation between x_2 and p_2 , with here $\Delta x_{1,2}^2 = \frac{1}{4} = \Delta p_{1,2}^2$ for the vacuum state [26,27].

Grangier and Grosshans elevate this concept of inference at a distance from the EPR analysis to “a criteria for non-separability [i.e., violation of Bell’s inequalities].” Specifically, they state that “the classical limit of no apparent violation of HI” [and hence the domain of local realism] is determined by the conditions

$$V_{x_2|x_1}V_{p_2|p_1} \geq \frac{1}{16}, \quad \text{and} \quad V_{x_1|x_2}V_{p_1|p_2} \geq \frac{1}{16}. \quad (14)$$

As shown in Refs. [26,27] for the states under consideration, the conditional variances of Eq. (13) are simply related to the following (unconditional) variances

$$\begin{aligned} \Delta x_{\mu_{ij}}^2 &= \langle (x_i - \mu_{ij}x_j)^2 \rangle, \\ \Delta p_{\nu_{ij}}^2 &= \langle (p_i - \nu_{ij}p_j)^2 \rangle. \end{aligned} \quad (15)$$

If we use a measurement of x_j to estimate x_i , then $\Delta x_{\mu_{ij}}^2$ is the variance of the error when the estimator is chosen to be $\mu_{ij}x_j$, and likewise for $\Delta p_{\nu_{ij}}^2$. For an optimal estimate, the parameters (μ_{ij}, ν_{ij}) are given by [26,27]

$$\mu_{ij}^{\text{opt}} = \frac{\langle x_i x_j \rangle}{\langle \Delta x_j^2 \rangle}, \quad \nu_{ij}^{\text{opt}} = \frac{\langle p_i p_j \rangle}{\langle \Delta p_j^2 \rangle}, \quad (16)$$

and in this case,

$$V_{x_i|x_j} = \Delta x_{\mu_{ij}^{\text{opt}}}^2, \quad \text{and} \quad V_{p_i|p_j} = \Delta p_{\nu_{ij}^{\text{opt}}}^2. \quad (17)$$

The “non-separability” condition of Grangier and Grosshans in Eq. (14) can then be re-expressed as

$$\Delta x_{\mu_{21}}^2 \Delta p_{\nu_{21}}^2 \geq \frac{1}{16}, \quad \text{and} \quad \Delta x_{\mu_{12}}^2 \Delta p_{\nu_{12}}^2 \geq \frac{1}{16}, \quad (18)$$

where we assume the optimized choice and drop the superscript ‘opt’. Again, Grangier and Grosshans take this condition of “no apparent violation of HI” as the operational signature of “nonseparability criteria” [violations of Bell inequalities], and hence, by their logic, to delineate the classical boundary for teleportation [11].

To make apparent the critical elements of the discussion, we next assume symmetric fluctuations as appropriate to the EPR state of Eq. (11), $\mu_{ij} = \mu_{ji} \equiv \mu$ and $\nu_{ij} = \nu_{ji} \equiv \nu$, with $\mu = -\nu$. Note that within the context of our simple model of the losses, the optimal value of μ is given by

$$\mu = \frac{\eta \sinh 2r}{(1 - \eta) + \eta \cosh 2r}, \quad (19)$$

where in the limit $r \gg 1$, $\mu \rightarrow 1$. For this case of symmetric fluctuations, the HI of Eq. (18) becomes

$$\Delta x_{\mu}^2 \Delta p_{\mu}^2 \geq \frac{1}{16}, \quad (20)$$

where

$$\begin{aligned} \Delta x_{\mu}^2 &= \langle (x_1 - \mu x_2)^2 \rangle = \langle (x_2 - \mu x_1)^2 \rangle, \\ \Delta p_{\mu}^2 &= \langle (p_1 + \mu p_2)^2 \rangle = \langle (p_2 + \mu p_1)^2 \rangle. \end{aligned} \quad (21)$$

Note that in general the inequality

$$V_1 V_2 \geq \frac{a^2}{4} \quad (22)$$

implies that

$$V_1 + V_2 \geq V_1 + \frac{a^2}{4V_1} \geq a, \quad (23)$$

so that the criterion of Eq. (20) for *classical* teleportation leads to

$$\Delta x_{\mu}^2 + \Delta p_{\mu}^2 \geq \frac{1}{2}. \quad (24)$$

Hence, the requirement of Grangier and Grosshans for *quantum* teleportation is that

$$\Delta x_{\mu}^2 + \Delta p_{\mu}^2 < \frac{1}{2}, \quad (25)$$

which for $r \gg 1$ becomes

$$\Delta x^2 + \Delta p^2 < \frac{1}{2}. \quad (26)$$

Here, $(\Delta x^2, \Delta p^2)$ are as defined in Eq. (21), now with $\mu = 1$;

$$\begin{aligned} \Delta x^2 &= \langle (x_1 - x_2)^2 \rangle, \\ \Delta p^2 &= \langle (p_1 + p_2)^2 \rangle, \end{aligned} \quad (27)$$

where from Eq. (11), we have that $\Delta x^2 + \Delta p^2 = \bar{\sigma}_-^2$ for the EPR beams (1,2). The claim of Grangier and Grosshans [11] is that the inequality of Eq. (20) serves as “the condition for no useful entanglement between the two beams,” where by ‘useful’ they refer explicitly to “the existence of quantum non-separability [violation of Bell’s inequalities].” The inequalities of Eqs. (18) and (20) are also related to criteria developed within the setting of quantum nondemolition detection (QND) [28], as discussed in the next section.

In a second paper [12], Grangier and Grosshans introduce an alternative criteria for the successful teleportation of coherent states, namely that

“the information content of the teleported quantum state is higher than the information content of any (classical or quantum) copy of the input state, that may be broadcasted classically.”

To quantify the concept of “information content” they introduce a “generalized fidelity” describing not the overlap of quantum states as is standard in the quantum information community, but rather the conditional probability $P(\alpha|I)$ that a particular coherent state $|\alpha\rangle$ was actually sent given “the available information I .” In effect, Grangier and Grosshans consider the following protocol. Victor sends to Alice some unknown coherent state $|\alpha_0\rangle$, with Alice making her best attempt to determine this state [34], and sending the resulting measurement outcome to Bob as in the standard protocol. Bob then does one of two things. In the first instance, he forwards only this classical message with Alice’s measurement outcome to Victor without reconstructing a quantum state. In the second case, he actually generates a quantum state conditioned upon Alice’s message and sends this state to Victor, who must then make his own measurement to deduce whether the teleported state corresponds to the one that he initially sent. For successful teleportation, Grangier and Grosshans demand that the information gained by Victor should be greater in the latter case where quantum states are actually generated by Bob than in the former case where only Alice’s classical measurement outcome is distributed. It is straightforward to show that Eq. (26) given above is sufficient to ensure that this second criteria is likewise satisfied for the teleportation of a coherent state $|\alpha\rangle$, albeit with the same caveat expressed in [10], namely that neither the set S of initial states $\{|\psi_{in}\rangle\}$ nor the distribution $P(|\psi_{in}\rangle)$ over these states is specified.

We now turn to an evaluation of these criteria of Grangier and Grosshans placing special emphasis on the issues of entanglement and violations of Bell’s inequalities, specifically because these are the concepts Grangier and Grosshans emphasize in their work [11,12].

IV. ENTANGLEMENT AND FIDELITY

A. Nonseparability of the EPR beams

To address the question of the nonseparability of the EPR beams, we refer to the papers of Duan *et al.* and of Simon [14,15], as well as related work by Tan [16]. For the definitions of (x_i, p_i) that we have chosen for the EPR beams (1,2), a sufficient condition for nonseparability (without an assumption of Gaussian statistics) is that

$$\Delta x^2 + \Delta p^2 < 1, \quad (28)$$

where Δx^2 and Δp^2 are defined in Eq. (27). This result follows from Eq. (3) of Duan *et al.* with $a = 1$ (and from a similar more general equation in Simon) [35]. Note that Duan *et al.* have $\Delta x_i^2 = \frac{1}{2} = \Delta p_i^2$ for the vacuum state, while our definitions lead to $\Delta x_i^2 = \frac{1}{4} = \Delta p_i^2$ for the

vacuum state, where for example, $\Delta x_1^2 = \langle x_1^2 \rangle$, and that all fields considered have zero mean.

Given the Wigner distribution $W_{\text{EPR}}^{\text{out}}$ as in Eq. (11), we find immediately that

$$\begin{aligned} \Delta x^2 + \Delta p^2 &= 2 \frac{\bar{\sigma}_-^2}{2} \\ &= \eta e^{-2r} + (1 - \eta)(1 + 2\bar{n}). \end{aligned} \quad (29)$$

For the case $\bar{n} = 0$, the resulting state is *always entangled for any $r > 0$ even for $\eta \ll 1$* , in agreement with the discussion in Duan *et al.* [14]. For nonzero \bar{n} , the state is entangled so long as

$$\bar{n} < \frac{\eta[1 - \exp(-2r)]}{2(1 - \eta)}. \quad (30)$$

We emphasize that in the experiment of Furusawa *et al.* [4] for which $\bar{n} = 0$ is the relevant case, the above inequality guarantees that teleportation was carried out with entangled (i.e., nonseparable) states for the EPR beams, independent of any assumption about whether these beams were Gaussian or pure states [36].

By contrast to the condition for entanglement given in Eq. (28), Grangier and Grosshans require instead the more stringent condition of Eq. (25) for successful teleportation. Although they would admit that the EPR beams are indeed entangled whenever Eq. (28) is satisfied,[†] they would term entanglement in the domain

$$\frac{1}{2} \leq \Delta x^2 + \Delta p^2 < 1$$

as not “useful” [11].

With regard to the QND-like conditions introduced by Grangier and Grosshans [11], we note that more general forms for the nonseparability condition of Eq. (28) are given in Refs. [14,15]. Of particular relevance is a condition for the variances of Eq. (15) for the case of symmetric fluctuations as for EPR state in Eq. (11), $\mu_{ij} = \mu_{ji} \equiv \mu$ and $\nu_{ij} = \nu_{ji} \equiv \nu$, with $\mu = -\nu$. Consider for example the first set of variances in Eq. (21), namely

$$\Delta x_\mu^2 = \langle (x_2 - \mu x_1)^2 \rangle \quad \text{and} \quad \Delta p_\mu^2 = \langle (p_2 + \mu p_1)^2 \rangle, \quad (31)$$

as would be appropriate for an inference of (x_2, p_2) from a measurement (at a distance) of (x_1, p_1) . In this case, a sufficient condition for entanglement of the EPR beams (1,2) may be obtained using Eq. (11) of Ref. [15] yielding

$$\Delta x_\mu^2 + \Delta p_\mu^2 < \frac{(1 + \mu^2)}{2}, \quad (32)$$

[†]Grangier was in fact unaware of Refs. [14,15] when Ref. [11] was originally posted, having had this work pointed out by us.

which reproduces Eq. (28) for $\mu = 1$. This equation for nonseparability implies that

$$\Delta x_\mu^2 \Delta p_\mu^2 < \frac{(1 + \mu^2)^2}{16}, \quad (33)$$

which is in the form of a Heisenberg-type inequality. Note that this inequality is satisfied for any $r > 0$ and $0 < \eta \leq 1$ for $\bar{n} = 0$. As discussed in Refs. [26,27], μ must be chosen in correspondence to the degree of correlation between the EPR beams, with $0 < \mu \leq 1$. An explicit expression for our current model given in Eq. (19). By contrast, in applying their QND-like conditions, Grangier and Grosshans demand to the contrary the Heisenberg-type inequality

$$\Delta x_\mu^2 \Delta p_\mu^2 < \frac{1}{16}. \quad (34)$$

Within the setting our current model, this condition can only be satisfied for efficiency $\eta > \frac{1}{2}$ [37]. Although this criterion has been found to be useful in the analysis of back-action evading measurement for quantum nondemolition detection, it apparently has no direct relevance to the question of entanglement, for $\mu = 1$ or otherwise.

Certainly, $\mu = 1$ is the case relevant to the actual teleportation protocol of Ref. [13]. However, Alice and Bob are surely free to explore the degree of correlation between their EPR beams and to test for entanglement by any means at their disposal, including simple measurements with $\mu \neq 1$.

Although the boundary expressed by the nonseparability conditions of Eqs. (28) and (32) are perhaps not so familiar in Quantum Optics, we stress that these criteria are associated quite directly with the standard condition for nonclassical behavior adopted by this community. Whenever Eqs. (28) and (32) are satisfied, the Glauber-Sudarshan phase-space function becomes non-positive [17], which for almost forty years has heralded entrance into a manifestly quantum or nonclassical domain. It is difficult to understand how Grangier and Grosshans propose to move from $\Delta x^2 + \Delta p^2 = 1$ to $\Delta x^2 + \Delta p^2 = \frac{1}{2}$ without employing quantum resources in the teleportation protocol (as is required when the Glauber-Sudarshan P -function is not positive definite). Their own work offers no suggestion of how this is to be accomplished.

B. Fidelity

Turning next to the question of the relationship of entanglement of the EPR beams [as quantified in Eq. (28)] to the fidelity attainable for teleportation *with these beams*, we recall from Eq. (2) of Ref. [4] that

$$F = \frac{1}{1 + \bar{\sigma}_-^2}, \quad (35)$$

where this result applies to teleportation of coherent states [38,39]. When combined with Eq. (29), we find that

$$F = \frac{1}{1 + (\Delta x^2 + \Delta p^2)}, \quad (36)$$

The criterion of Eq. (28) for nonseparability then guarantees that nonseparable EPR states as in Eqs. (4,11) (be they mixed or pure) are sufficient to achieve

$$F > F_{\text{classical}} = \frac{1}{2}, \quad (37)$$

whereas separable states must have $F \leq F_{\text{classical}} = \frac{1}{2}$, although we emphasize that this bound applies for the average fidelity for coherent states distributed over the entire complex plane [6,39].

We thereby demonstrate that the condition $F > F_{\text{classical}} = \frac{1}{2}$ for quantum teleportation as established in Ref. [6] coincides with that for nonseparability (i.e., entanglement) of Refs. [14,15] for the EPR state of Eq. (11). Note that for $\bar{n} = 0$, we have

$$F = \frac{1}{2 - \eta(1 - e^{-2r})}, \quad (38)$$

so that the entangled EPR beams considered here (as well as in Refs. [11,12]) provide a sufficient resource for beating the limit set by a classical channel alone for any $r > 0$, so long as $\eta > 0$. In fact, the quantities $(\Delta x^2, \Delta p^2)$ are readily measured experimentally, so that the entanglement of the EPR beams can be operationally verified, as was first accomplished in Ref. [26], and subsequently in Ref. [4]. We stress that independently of any further assumption, the condition of Eq. (28) is sufficient to ensure entanglement for pure or mixed states [40,41].

The dependence of fidelity F on the degree of squeezing r and efficiency η as expressed in Eq. (38) is illustrated in Figure 1. Here, in correspondence to an experiment with fixed overall losses and variable parametric gain in the generation of the EPR entangled state, we show a family of curves in Figure 1 each of which is drawn for constant η as a function of r . Clearly, $F > F_{\text{classical}} = \frac{1}{2}$ and hence nonseparability results in each case. The only apparent significance of $F = \frac{2}{3}$ as championed by Grangier and Grosshans (and which results for $\Delta x^2 + \Delta p^2 = \frac{1}{2}$) is to bound F for $\eta = 0.5$.

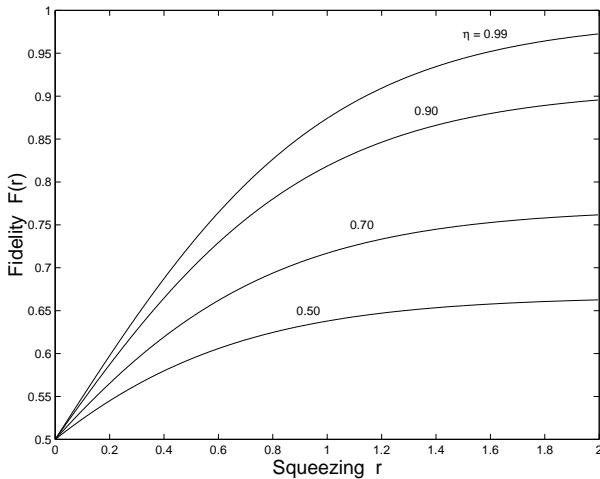


FIG. 1. Fidelity F as given by Eq. (38) versus the degree of squeezing r for fixed efficiency η . From top to bottom, the curves are drawn with $\eta = \{0.99, 0.90, 0.70, 0.50\}$ in correspondence to increasing loss $(1 - \eta)$. Note that $F_{\text{classical}} = \frac{1}{2}$ provides a demarcation between separable and nonseparable states (mixed or otherwise), while $F = \frac{2}{3}$ is apparently of no particular significance, the contrary claims of Ref. [11,12] notwithstanding. Note that for $\eta = 1$, $r = \frac{\ln 2}{2} = 0.3466$ gives $F = \frac{2}{3}$, corresponding to -3dB of squeezing. In all cases, $\eta = 0$.

As for the criterion of “information content” introduced by Grangier and Grosshans [12], we note it can be easily understood from the current analysis and the original discussion in Ref. [13]. Each of the interventions by Alice and Bob represent one unit of added vacuum noise that will be convolved with the initial input state in the teleportation protocol (the so-called *quduties*). Grangier and Grosshans compare the following two situations: (i) Bob passes directly the classical information that he receives to Victor and (ii) Bob generates a quantum state in the usual fashion that is then passed to Victor. Grangier and Grosshans would demand that Victor should receive the same information in these two cases, which requires that $\bar{\sigma}_-^2 = \Delta x^2 + \Delta p^2 < \frac{1}{2}$, and hence $F > \frac{2}{3}$. That is, as the degree of correlation between the EPR beams is increased, there comes a point for which $\Delta x^2 + \Delta p^2 = \frac{1}{2}$, and for which each of Alice and Bob’s excess noise has been reduced from 1 quduty each to $\frac{1}{2}$ quduty each. At this point, Grangier and Grosshans would (arbitrarily) assign the entire resulting noise of $\frac{1}{2} + \frac{1}{2} = 1$ quduties to Alice, with then the perspective that Bob’s state recreation adds no noise. Of course one could equally well make the complementary assignment, namely 1 quduty to Bob and none to Alice (again in the case with $\bar{\sigma}_-^2 = \frac{1}{2}$). The point that seems to be missed by Grangier and Grosshans is that key to quantum teleportation is the transport of quantum states. Although they correctly state that “there is *no* extra noise associated to the reconstruction: given a measured β , one can exactly reconstruct the coherent state $|\beta\rangle$, by using

a deterministic translation of the vacuum.” Bob can certainly make such a state deterministically, but it is an altogether different matter for Victor to receive a classical number from Bob in case (i) as opposed to the actual quantum state in (ii). In this latter case apart from having a physical state instead of a number, Victor must actually make his own measurement with the attendant uncertainties inherent in $|\beta\rangle$ then entering. Analogously, transferring measurement results about a qubit, without recreating a state at the output (i.e., without sending an actual *quantum state* to Victor), is not what is normally considered to constitute quantum teleportation relative to the original protocol of Bennett *et al.* [1].

Turning next to the actual experiment of Ref. [4], we note that a somewhat subtle issue is that the detection efficiency for Alice of the unknown state was not 100%, but rather was $\eta_A^2 = 0.97$. Because of this, the fidelity for classical teleportation (i.e., with vacuum states in place of the EPR beams) did not actually reach $\frac{1}{2}$, but was instead $F_0 = 0.48$. This should not be a surprise, since there is nothing to ensure that a given classical scheme will be optimal and actually reach the bound $F_{\text{classical}} = \frac{1}{2}$. Hence, the starting point in the experiment with $r = 0$ had $F_0 < F_{\text{classical}}$; the EPR beams with $r > 0$ (which were in any event entangled by the above inequality) then led to increases in fidelity from F_0 upward, exceeding the classical bound $F_{\text{classical}} = \frac{1}{2}$ for a small (but not infinitesimal) degree of squeezing. Note that the whole effect of the offset $F_0 = 0.48 < \frac{1}{2}$ can be attributed to the lack of perfect (homodyne) efficiency at Alice’s detector for the unknown state. In the current discussion for determining the classical bound in the *optimal* case, we set Alice’s detection efficiency instead to $\eta_A^2 = 1$, then as shown above, classical teleportation will achieve $F = \frac{1}{2}$.

Independent of such considerations, we reiterate that the nonseparability condition of Refs. [14,15] applied to the EPR state of Eqs. (4) and (11) leads to the same result $F_{\text{classical}} = \frac{1}{2}$ [Eqs. (36) and (37)] as did our previous analysis based upon teleportation with only a classical communication channel linking Alice and Bob [6]. This convergence further supports $F_{\text{classical}} = \frac{1}{2}$ as the appropriate quantum-classical boundary for the teleportation of coherent states, the claims of Grangier and Grosshans notwithstanding. Relative to the original work of Bennett *et al.* [1], exceeding the bound $F_{\text{classical}} = \frac{1}{2}$ for the teleportation of coherent can be accomplished with a classical channel and entangled (i.e., nonseparable) EPR states, be they mixed or pure, as is made clear by the above analysis and as has been operationally confirmed [4].

We should however emphasize that the above conclusions concerning nonseparability and teleportation fidelity apply to the specific case of the EPR state as in Eq. (11), for which inequality Eq. (28) represents both a necessary and sufficient criterion for nonseparability ac-

cording to Refs. [14,15]. More generally, for arbitrary entangled states, nonseparability does not necessarily lead to $F > \frac{1}{2}$ in coherent-state teleportation [40,41].

V. BELL'S INEQUALITIES

The papers by Banaszek and Wodkiewicz [18,19] provide our point of reference for a discussion of Bell's inequalities. In these papers, the authors introduce an appropriate set of measurements that lead to a Bell inequality of the CHSH type. More explicitly, Eq.(4) of Ref. [18] gives the operator $\hat{\Pi}(\alpha; \beta)$ whose expectation values are to be measured. Banaszek and Wodkiewicz point out that the expectation value of $\hat{\Pi}(\alpha; \beta)$ is closely related to the Wigner function of the field being investigated, namely

$$W(\alpha; \beta) = \frac{4}{\pi^2} \Pi(\alpha; \beta), \quad (39)$$

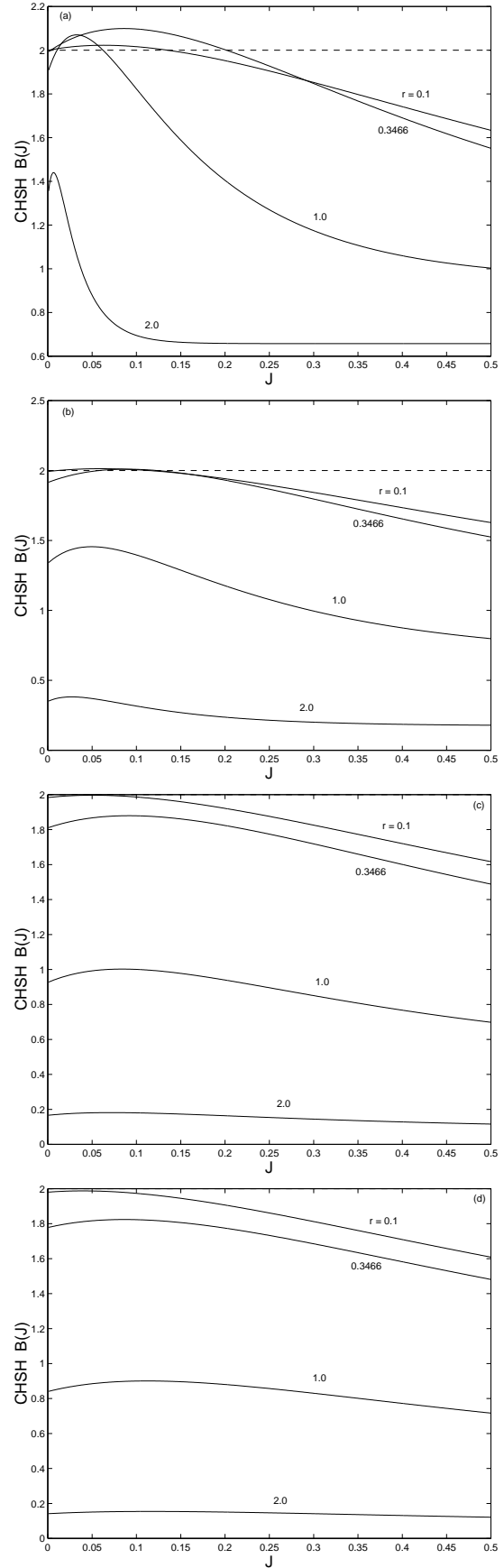
where $\Pi(\alpha; \beta) = \langle \hat{\Pi}(\alpha; \beta) \rangle$.

For the entangled state shared by Alice and Bob in the teleportation protocol, we identify $W_{\text{EPR}}^{\text{out}}$ as the relevant Wigner distribution for the modes (1,2) of interest, so that

$$\begin{aligned} & \Pi_{\text{EPR}}^{\text{out}}(x_1, p_1; x_2, p_2) \\ &= \frac{1}{\bar{\sigma}_+^2 \bar{\sigma}_-^2} \exp\left\{ -[(x_1 + x_2)^2 + (p_1 - p_2)^2] / \bar{\sigma}_+^2 \right. \\ & \quad \left. - [(x_1 - x_2)^2 + (p_1 + p_2)^2] / \bar{\sigma}_-^2 \right\}. \end{aligned} \quad (40)$$

Banaszek and Wodkiewicz show that $\Pi_{\text{EPR}}^{\text{out}}(x_1, p_1; x_2, p_2)$ gives directly the correlation function that would otherwise be obtained from a particular set of observations over an ensemble representing the field with density operator $\hat{\rho}$, where the actual measurements to be made are as described in Refs. [18,19]. In simple terms, $\hat{\Pi}_{\text{EPR}}^{\text{out}}(0, 0; 0, 0)$ is the parity operator for separate measurements of photon number on modes (1,2), with then nonzero (x_i, p_i) corresponding to a “rotation” on the individual mode i that precedes its parity measurement.

FIG. 2. The function $\mathcal{B}(\mathcal{J})$ from Eq. (41) as a function of \mathcal{J} for various values of (r, η) . Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. In each of the plots (a)-(d) a family of curves is drawn for fixed efficiency η and four values of $r = \{0.1, \frac{\ln 2}{2}, 1.0, 2.0\}$. (a) $\eta = 0.99$, (b) $\eta = 0.90$, (c) $\eta = 0.70$, (d) $\eta = 0.50$; in all cases, $\bar{n} = 0$.



The function constructed by Banaszek and Wodkiewicz to test for local hidden variable theories is denoted by \mathcal{B} and is defined by

$$\begin{aligned} \mathcal{B}(\mathcal{J}) &= \Pi_{\text{EPR}}^{\text{out}}(0, 0; 0, 0) + \Pi_{\text{EPR}}^{\text{out}}(\sqrt{\mathcal{J}}, 0; 0, 0) \\ &+ \Pi_{\text{EPR}}^{\text{out}}(0, 0; -\sqrt{\mathcal{J}}, 0) - \Pi_{\text{EPR}}^{\text{out}}(\sqrt{\mathcal{J}}, 0; -\sqrt{\mathcal{J}}, 0), \end{aligned} \quad (41)$$

where \mathcal{J} is a positive (real) constant. As shown in Ref. [18,19], any local theory must satisfy

$$-2 \leq \mathcal{B} \leq 2. \quad (42)$$

As emphasized by Banaszek and Wodkiewicz for the lossless case, $\Pi_{\text{EPR}}^{\text{out}}(0, 0; 0, 0) = 1$ “describes perfect correlations ... as a manifestation of ... photons always generated in pairs.”

There are several important points to be made about this result. In the first place, in the ideal case with no loss ($\eta = 1$), there is a violation of the Bell inequality of Eq. (42) for any $r > 0$. Further, this threshold for the onset of violations of the CHSH inequality coincides with the threshold for entanglement as given in Eq. (28), which likewise is the point for surpassing $F_{\text{classical}} = \frac{1}{2}$ as in Eqs. (36,37) and as shown in our prior analysis of Ref. [6] which is notably based upon a quite different approach.

Significantly, there is absolutely nothing special about the point $r = \frac{\ln 2}{2} \approx 0.3466$ (i.e., the point for which $\exp[-2r] = 0.5$ and for which $F = \frac{2}{3}$ for the teleportation of coherent states), in contradistinction to the claims of Grangier and Grosshans to the contrary [11,12]. Instead, any $r > 0$ leads to a nonseparable EPR state, to a violation of a Bell inequality, and to $F > F_{\text{classical}} = \frac{1}{2}$ for the teleportation of coherent states. There is certainly no surprise here since we are dealing with pure states for $\eta = 1$ [42].

We next examine the case with $\eta < 1$, which is clearly of interest for any experiment. Figure 2 illustrates the behavior of \mathcal{B} as a function of \mathcal{J} for various values of the squeezing parameter r and of the efficiency η . Note that throughout our analysis in this section, we make no attempt to search for optimal violations, but instead follow dutifully the protocol of Banaszek and Wodkiewicz as expressed in Eq. (41) for the case with losses as well.

From Figure 2 we see that for any particular set of parameters (r, η) , there is an optimum value \mathcal{J}_{max} that leads to a maximum value for $\mathcal{B}(\mathcal{J}_{\text{max}})$, which is a situation analogous to that found in the discrete variable case. By determining the corresponding value \mathcal{J}_{max} at each (r, η) , in Figure 3 we construct a plot that displays the dependence of \mathcal{B} on the squeezing parameter r for various values of efficiency η . Note that all cases shown in the figure lead to fidelity $F > F_{\text{classical}}$.

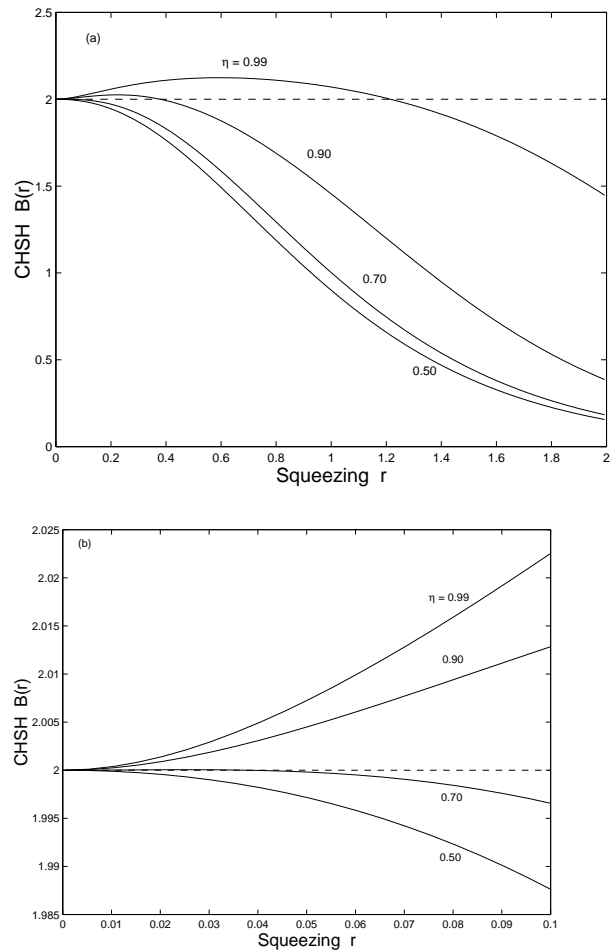


FIG. 3. (a) The quantity \mathcal{B} from Eq. (41) as a function of r for various values of efficiency $\eta = \{0.99, 0.90, 0.70, 0.50\}$ as indicated. At each point in (r, η) , the value of \mathcal{J} that maximizes \mathcal{B} has been chosen. Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. Also note that $F > \frac{1}{2}$ for all $r > 0$. (b) An expanded view of \mathcal{B} in the small r region $r \leq 0.1$. Note that in the case $\eta = 0.70$, $\mathcal{B} > 2$ for small r . In all cases, $\bar{n} = 0$.

For $\frac{2}{3} < \eta \leq 1$ there are regions in r that produce direct violations of the Bell inequality considered here, namely $\mathcal{B} > 2$ [43]. In general, these domains with $\mathcal{B} > 2$ contract toward smaller r with increasing loss $(1 - \eta)$. In fact as r increases, η must become very close to unity in order to preserve the condition $\mathcal{B} > 2$, where for $r \gg 1$,

$$2(1 - \eta) \cosh(2r) \ll 1. \quad (43)$$

This requirement is presumably associated with the EPR state becoming more “nonclassical” with increasing r and hence more sensitive to dissipation [44]. Stated somewhat more quantitatively, recall that the original state $|EPR\rangle_{1,2}$ of Eq. (3) is expressed as a sum over correlated photon numbers for each of the two EPR beams (1, 2). The determination of \mathcal{B} derives from (displaced) parity measurements on the beams (1, 2) (i.e., projections onto odd and even photon number), so that \mathcal{B} should be sen-

sitive to the loss of a single photon. The mean photon number \bar{n}_i for either EPR beam goes as $\sinh^2 r$, with then the probability of losing no photons after encountering the beam-splitter with transmission η scaling as roughly $p_0 \sim [\eta]^{\bar{n}_i}$. We require that the total probability for the loss of one or more photons to be small, so that

$$(1 - p_0) \ll 1, \quad (44)$$

and hence for $(1 - \eta) \ll 1$ and $r \gg 1$ that

$$(1 - \eta)\bar{n}_i \sim (1 - \eta)\exp(2r) \ll 1, \quad (45)$$

in correspondence to Eq. (43) [45].

On the other hand, note that small values of r in Figure 3 lead to direct violations of the CHSH inequality $\mathcal{B} > 2$ with much more modest efficiencies [44]. In particular, note that for $r = \frac{\ln 2}{2} \approx 0.3466$ and $\eta = 0.90$, $F < \frac{2}{3}$ [from Eq. (38)]. This case and others like it provide examples for which mixed states are nonseparable and yet directly violate a Bell inequality, but for which $F \leq \frac{2}{3}$. Such mixed states do not satisfy the criteria of Grangier and Grosshans (neither with respect to their Heisenberg-type inequality nor with respect to their information exchange), yet they are states for which $\frac{1}{2} < F \leq \frac{2}{3}$ and $\mathcal{B} > 2$, which in and of itself calls the claims of Grangier and Grosshans into question. There remains the possibility that $F > \frac{2}{3}$ might be sufficient to warranty that mixed states in this domain would satisfy that $\mathcal{B} > 2$, and hence to exclude a description of the EPR state in terms of a local hidden variables theory.

To demonstrate that this is emphatically not the case, we examine further the relationship between the quantity \mathcal{B} relevant to the CHSH inequality and the fidelity F . Figure 4 shows a parametric plot of \mathcal{B} versus F for various values of the efficiency η . The curves in this figure are obtained from plots as in Figures 1 and 3 by eliminating the common dependence on r . From Figure 4, we are hard pressed to find any indication that the value $F = \frac{2}{3}$ is in any fashion noteworthy with respect to violations of the CHSH inequality. In particular, for efficiency $\eta \simeq 0.90$ most relevant to current experimental capabilities, the domain $F > \frac{2}{3}$ is one largely devoid of instances with $\mathcal{B} > 2$, in contradistinction to the claim of Grangier and Grosshans that this domain is somehow “safer” [11] with respect to violations of Bell’s inequalities. Moreover, contrary to their dismissal of the domain $\frac{1}{2} < F \leq \frac{2}{3}$ as not being manifestly quantum, we see from Figure 4 that there are in fact regions with $\mathcal{B} > 2$. Overall, the conclusions of Grangier and Grosshans [11] related to the issues of violation of a Bell inequality and of teleportation fidelity are simply not supported by an actual quantitative analysis.

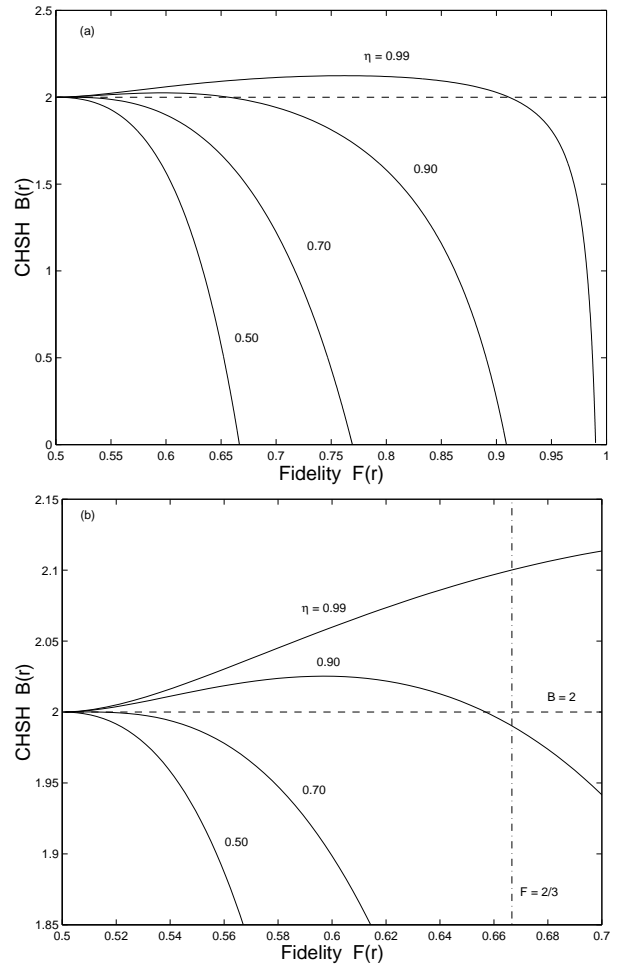


FIG. 4. (a) A parametric plot of the CHSH quantity \mathcal{B} [Eq. (41)] versus fidelity F [Eq. (38)]. The curves are constructed from Figures 1 and 3 by eliminating the r dependence, now over the range $0 \leq r \leq 5$, with r increasing from left to right for each trace. The efficiency η takes on the values $\eta = \{0.99, 0.90, 0.70, 0.50\}$ as indicated; in all cases, $\bar{n} = 0$. Recall that $\mathcal{B} > 2$ heralds a direct violation of the CHSH inequality, with the dashed line $\mathcal{B} = 2$ shown. (b) An expanded view around $\mathcal{B} = 2$. Note that $\mathcal{B} > 2$ is impossible for $F \leq F_{\text{classical}} = \frac{1}{2}$, but that $\mathcal{B} > 2$ for $F > F_{\text{classical}}$ in various domains (including for $\eta = 0.70$ at small r). The purported boundary $F = \frac{2}{3}$ proposed by Grangier and Grosshans [11,12] is seen to have no particular significance. Contrary to their claims, $F = \frac{2}{3}$ provides absolutely no warranty that $\mathcal{B} > 2$ for $F > \frac{2}{3}$, nor does it preclude $\mathcal{B} > 2$ for $F < \frac{2}{3}$.

While the above results follow from the particular form of the CHSH inequality introduced by Banaszek and Wodkiewicz [18,19], we should note that another quite different path to a demonstration of the inadequacy of local realism for continuous quantum variables has recently been proposed by Ralph, Munro, and Polkinghorne [46]. These authors consider a novel scheme involving measurements of quadrature-phase amplitudes for two entangled beams (A, B). These beams are formed by combining *two* EPR states (i.e., a total of four modes, two for

each beam). Relevant to our discussion is that maximal violations of a CHSH inequality (i.e., $\mathcal{B} = 2\sqrt{2}$) are predicted for $r \ll 1$, with then a decreasing maximum value of \mathcal{B} for increasing r . Once again, the threshold for onset of the violation of a Bell's inequality coincides with the threshold for entanglement of the relevant fields [i.e., Eq. (28)], with no apparent significance to the boundary set by the Heisenberg-type inequality Eq. (26) of Grangier and Grosshans.

To conclude this section, we would like to inject a note of caution concerning any discussion involving issues of testing Bell's inequalities and performing quantum teleportation. We have placed them in juxtaposition here to refute the claims of Grangier and Grosshans related to a possible connection between the bound $F = \frac{2}{3}$ and violation of Bell's inequalities (here, via the behavior of the CHSH quantity \mathcal{B}). However, in our view there is a conflict between these concepts, with an illustration of this point provided by the plot of the CHSH quantity \mathcal{B} [Eq. (42)] versus fidelity F [Eq. (38)] in Figure 4. For example, for $\eta = 0.90$, $\mathcal{B} > 2$ over the range $0.50 < F \lesssim 0.66$, while $\mathcal{B} < 2$ for larger values of F . Hence, local hidden variables theories are excluded for modest values of fidelity $0.50 < F \lesssim 0.66$, but not for larger values $F \gtrsim 0.66$. This leads to the strange conclusion that quantum resources are required for smaller values of fidelity but not for larger ones. The point is that the nonseparable states that can enable quantum teleportation, can *in a different context* also be used to demonstrate a violation of local realism. Again, the juxtaposition of these concepts in this section is in response to the work of Ref. [11], which in any event offers no quantitative evidence in support of their association.

VI. BELL'S INEQUALITIES FOR SCALED CORRELATIONS

The conclusions reached in the preceding section about violations of the CHSH inequality by the EPR (mixed) state for modes (1, 2) follow directly from the analysis of Banaszek and Wodkiewicz [18,19] as extended to account for losses in propagation. Towards the end of making these results more amenable to experimental investigation, recall that the more traditional versions of the Bell inequalities formulated for spin $\frac{1}{2}$ particles or photon polarizations are based upon an analysis of the expectation value

$$E(\vec{a}, \vec{b}) \quad (46)$$

for detection events at locations (1, 2) with analyzer settings along directions (\vec{a}, \vec{b}) . As emphasized by Clauser and Shimony, actual experiments do not measure directly $E(\vec{a}, \vec{b})$ but rather record a reduced version due to “imperfections in the analyzers, detectors, and state

preparation [20].” Even after more than thirty years of experiments, no *direct* violation of the CHSH inequality has been recorded, where by *direct* we mean without the need for post-selection to compensate for propagation and detection efficiencies (also called *strong violations*) [22,23]. Rather, only subsets of events that give rise to coincidences are included for various polarization settings. This “problem” is the so-called detector efficiency loophole that several groups are actively working to close.

Motivated by these considerations, we point out that an observation of violation of a Bell-type inequality has recently been reported [24], based in large measure upon the earlier proposal of Ref. [47], as well as that of Refs. [18,19]. This experiment was carried out in a pulsed mode, and utilized a source that generates an EPR state of the form given by Eq. (11) in the limit $r \ll 1$. Here, the probability $P(\alpha_1, \alpha_2)$ of detecting a coincidence event between detectors (D_1, D_2) for the EPR beams (1, 2) is given by

$$P(\alpha_1, \alpha_2) = M[1 + V \cos(\phi_1 - \phi_2 + \theta)], \quad (47)$$

with then the correlation function E relevant to the construction of a CHSH inequality $-2 \leq S \leq 2$ given by

$$E(\phi_1, \phi_2) = V \cos(\phi_1 - \phi_2 + \theta), \quad (48)$$

where the various quantities are as defined in association with Eqs. (2,3) in Ref. [24]. Note that the quantity M represents an overall scaling that incorporates losses in propagation and detection. Significantly, Kuzmich *et al.* demonstrated a violation of a CHSH inequality ($S_{\text{exp}} = 2.46 \pm 0.06$) in the limit $r \ll 1$ and with inefficient propagation and detection $\eta \ll 1$, albeit with the so-called “detection” or “fair-sampling” loophole.

In terms of our current discussion, this experimental violation of a CHSH inequality is only just within the nonseparability domain $\Delta x^2 + \Delta p^2 < 1$ (by an amount that goes as $\eta r \ll 1$), yet it generates a large violation of a CHSH inequality. If this same EPR state were employed for the teleportation of coherent states, the fidelity obtained would likewise be only slightly beyond the quantum-classical boundary $F_{\text{classical}} = \frac{1}{2}$. It would be far from the boundary $F = \frac{2}{3}$ offered by Grangier and Grosshans as the point for “useful entanglement,” yet it would nonetheless provide an example of teleportation with fidelity $F > \frac{1}{2}$ and of a violation of a CHSH inequality. Of course, the caveat would be the aforementioned “fair-sampling” loophole, but this same restriction accompanies all previous experimental demonstrations of violations of Bell's inequalities. Once again, we find no support for the purported significance of the criteria offered by Grangier and Grosshans [11,12].

VII. CONCLUSIONS

Beyond the initial analysis of Ref. [6], we have examined further the question of the appropriate point of demarcation between the classical and quantum domains for the teleportation of coherent states. In support of our previous result that fidelity $F_{\text{classical}} = \frac{1}{2}$ represents the bound attainable by Alice and Bob if they make use only of a classical channel, we have shown that the nonseparability criteria introduced in Refs. [14,15] are sufficient to ensure fidelity beyond this bound for teleportation with the EPR state of Eq. (11), which is in general a mixed state. Significantly, the threshold for entanglement for the EPR beams as quantified by these nonseparability criteria coincides with the standard boundary between classical and quantum domains employed in Quantum Optics, namely that the Glauber-Sudarshan phase-space function becomes non-positive definite [17].

Furthermore, we have investigated possible violations of Bell's inequalities and have shown that the threshold for the onset of such violations again corresponds to $F_{\text{classical}} = \frac{1}{2}$. For thermal photon number $\bar{n} = 0$ as appropriate to current experiments, direct violations of a CHSH inequality are obtained over a large domain in the degree of squeezing r and overall efficiency η . Significant relative to the claims of Grangier and Grosshans [11,12] is that there is a regime for nonseparability and violation of the CHSH inequality for which $F < \frac{2}{3}$ and for which their Heisenberg inequalities are not satisfied. Moreover, the experiment of Ref. [24] has demonstrated a violation of the CHSH inequality in this domain for $(r, \eta) \ll 1$ (i.e., F would be only slightly beyond $\frac{1}{2}$), albeit with the caveat of the "fair-sampling" loophole. We conclude that fidelity $F > \frac{2}{3}$ offers absolutely no warranty or "safety" relative to the issue of violation of a Bell inequality (as might be desirable, for example, in quantum cryptography), in direct disagreement with the assertions by Grangier and Grosshans. Quite the contrary, larger r (and hence larger F) leads to an exponentially decreasing domain in allowed loss $(1 - \eta)$ for violation of the CHSH inequality, as expressed by Eq. (43) [45].

Moreover, beyond the analysis that we have presented here, there are several other results that support $F_{\text{classical}} = \frac{1}{2}$ as being the appropriate boundary between quantum and classical domains. In particular, we note that any nonseparable state and hence also our mixed EPR state is always capable of teleporting perfect entanglement, i.e., one half of a pure maximally entangled state. This applies also to those nonseparable states which lead to fidelities $\frac{1}{2} < F \leq \frac{2}{3}$ in coherent-state teleportation. According to Refs. [11,12], this would force the conclusion that there is entanglement that is capable of teleporting truly nonclassical features (i.e., entanglement), but which is not "useful" for teleporting rather more classical states such as coherent states. Further,

in Ref. [48] it has been shown that entanglement swapping can be achieved with two pure EPR states for *any nonzero squeezing* in both initial states. Neither of the initial states has to exceed a certain amount of squeezing in order to enable successful entanglement swapping. This is another indication that $F = \frac{2}{3}$, which is exceedable in coherent-state teleportation only with more than 3 dB squeezing, is of no particular significance.

We also point out that Giedke et al. have shown that for all bipartite Gaussian states, nonseparability implies distillability [49,50]. This result applies to those nonseparable states for which $\frac{1}{2} < F \leq \frac{2}{3}$ in coherent state teleportation, which are otherwise dismissed by Grangier and Grosshans as not "useful." To the contrary, entanglement distillation could be applied to the mixed EPR states employed for teleportation in this domain (and in general for $F > \frac{1}{2}$) [51], leading to enhanced teleportation fidelities and to expanded regions for violations of Bell's inequalities for the distilled subensemble.

By contrast, there appears to be no support for the claims of Grangier and Grosshans [11,12] that their so-called Heisenberg inequality and information exchange are somehow "special" with respect to the issues of separability and violations of Bell's inequalities. They have neither found fault in the prior analysis of Ref. [6], nor with the application of the work on nonseparability [14–16] to the current problem. They have likewise provided no analysis that directly supports their assertion that their Heisenberg inequality is in any way significant to the possibility that "the behavior of the *observed* quantities can be mimicked by a *classical* and *local* model." [11] Rather, they attempt to set aside by *fiat* a substantial body of evidence in favor of the boundary $F_{\text{classical}} = \frac{1}{2}$ for the teleportation of coherent states with a lack of rigor indicated by their claim that " $F = \frac{2}{3}$ would be much safer." [11]

However, having said this, we emphasize that there is no criterion for quantum teleportation that is sufficient to all tasks. For the special case of teleportation of coherent states, the boundary between classical and quantum teleportation is fidelity $F_{\text{classical}} = \frac{1}{2}$, as should by now be firmly established. Fidelity $F > \frac{2}{3}$ will indeed enable certain tasks to be accomplished that could not otherwise be done with $\frac{1}{2} < F \leq \frac{2}{3}$. However, $F = \frac{2}{3}$ is in no sense an important point of demarcation for entrance into the quantum domain. There is instead a hierarchy of fidelity thresholds that enable ever more remarkable tasks to be accomplished via teleportation within the quantum domain, with no one value being sufficient for all possible purposes. For example, if the state to be teleported were some intermediate result from a large-scale quantum computation as for Shor's algorithm, then surely the relevant fidelity threshold would be well beyond any value currently accessible to experiment, $F \sim 1 - \epsilon$, with $\epsilon \lesssim 10^{-4}$ to be compatible with current work in fault tolerant architectures. We have never claimed that $F = \frac{1}{2}$

endows special powers for all tasks such as these, only that it provides an unambiguous point of entry into the quantum realm for the teleportation of coherent states.

HJK gratefully acknowledges critical input from A. C. Doherty, H. Mabuchi, E. S. Polzik, and J. P. Preskill, and support from the NSF (Grant No. PHY-9722674), the Institute for Quantum Information (IQI) funded by the NSF-ITR Program, and the ONR. SLB and PvL are funded in part under project QUICOV as part of the IST-FET-QJPC programme. PvL acknowledges support by a DAAD Doktorandenstipendium (HSP III)

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 - [36] For the experiment of Ref. [4], the measured variances were $\Delta x^2 \approx (0.8 \times \frac{1}{2}) \approx \Delta p^2$, so that $\Delta x^2 + \Delta p^2 \approx 0.8 < 1$.
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 - [39] The expressions of Eqs. (32-35) are strictly applicable only for the case gain $g = 1$ for teleportation of coherent states uniformly distributed over the entire complex plane. More generally, when working with a restricted alphabet of states (e.g., coherent amplitudes selected from a Gaussian distribution), the optimal gain is not unity when referenced to the fidelity averaged over the input alphabet. In fact as shown in Ref. [6], the optimal gain is $g = 1/(1 + \lambda)$ for an input alphabet of coherent states distributed according to $p(\beta) = (\lambda/\pi) \exp(-\lambda|\beta|^2)$. When incorporated into the current analysis, we find that nonseparable EPR states are sufficient to achieve $F > (1 + \lambda)/(2 + \lambda)$ (again with optimal gain $g \neq 1$) although F is no longer a monotonic function of r as

- in Figure 1. This result is in complete correspondence with our prior result that $F_{\text{classical}} = (1 + \lambda)/(2 + \lambda)$ is the bound for teleportation when only a classical channel is employed. To simplify the discussion, here we set $\lambda = 0$ throughout, with then the optimal gain $g = 1$ and $F_{\text{classical}} = \frac{1}{2}$.
- [40] Note that for the noisy EPR states considered here with Gaussian statistics, Eqs. (36) and (37) are both necessary and sufficient conditions. However, this is not generally the case for states with Gaussian statistics; there are in fact nonseparable Gaussian states for which Eq. (28) is not true. Close examination of Ref. [14] shows that their results concerning necessary and sufficient conditions for states with Gaussian statistics apply only to Gaussian states in a particular standard form. Explicit counter examples can be constructed from Ref. [41].
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